PRAMANA
—journal of
physics

Vol. 48, No. 1,
January 1997
pp. 99–108

Chaos in modulated logistic systems

V M NANDAKUMARAN
International School of Photonics, Cochin University of Science and Technology, Cochin 682 022, India

Abstract. This paper is a review of the work done on the dynamics of modulated logistic systems. Three different problems are treated, viz, the modulated logistic map, the parametrically perturbed logistic map and the combination map obtained by combining two maps of the quadratic family. Many of the interesting features displayed by these systems are discussed.

Keywords. One dimensional maps; universality; parametric perturbation; scaling relations.

PACS Nos 05.40; 05.45; 47.52

1. Introduction

The complicated dynamical behaviour that can arise in the low dimensional deterministic nonlinear systems has generated a lot of intense research in the last two decades [1, 2]. These studies have shown that simple physical systems obeying simple laws can often show complex and chaotic behaviour. Since several nonlinear systems can be analysed using one dimensional maps, such maps have come to play an important role in understanding many of the general features of chaotic systems. One of the simplest one dimensional maps, namely the logistic map, originally introduced in the study of the population dynamics of non overlapping generations in biology [3] has become a paradigm for such systems. The logistic map is a one-hump map of the form

$$X_{n+1} = 4\lambda X_n(1 - X_n) \tag{1}$$

defined in the interval (0, 1) and is characterized by a control parameter λ which is also varied in the interval (0, 1). This system undergoes a transition to chaos via the well-known period doubling route and shows universal metric [4] as well as structural properties [5].

One of the earliest experimental observations of the period doubling route to chaos was in CO_2 laser [6]. However, in many of the experimental situations the control parameter can have a time dependence of its own. Therefore, it would be interesting to consider the dynamics of a map with a time-dependent control parameter.

In one of the earlier studies, Kapral and Mandel [7] investigated a non autonomous quadratic map wherein the control parameter was assumed to vary linearly with time. They have observed that this time dependence of the parameter delayed the onset of bifurcations in the system. The system also showed bistability and hysteresis. Ruelle had suggested the study of dynamical systems with adiabatically fluctuating parameters where the control parameter has a very slow variation in time and this time dependence itself might be determined by a dynamical system [8].

In this paper the dynamics of the modulated logistic system is presented. We use the term modulated logistic system (MLS) to describe the following three situations:

(i) A logistic map in which the control parameter has a discrete time-dependence determined by a dynamical system – a modulated logistic map (MLM).
(ii) A logistic map wherein the control parameter is perturbed by the addition of a periodic term (parametrically perturbed logistic map).
(iii) A combination map where the logistic map is combined with another map of the same quadratic family.

The paper is organized as folllows. In § 2, we introduce the MLM and present some of the earlier results of Harikrishnan and Nandakumaran [9–12]. Some of the related works in this field are also discussed in the section [13]. We assume that the time evolution of the control parameter is determined by another logistic map. This coupled system shows many interesting features such as the locking of the periodicities of the two subsystems. Section 3 deals with the parametrically perturbed logistic map. Here the control parameter of the logistic map is perturbed by a sequence of periodic pulses. Such a system is relevant for modelling those systems which are subjected to periodic stimuli. The dynamics of the system shows many interesting and novel features such as the formation of bubble structure in the bifurcation diagram, emergence of periodic behaviour after a chaotic regime etc. The combination map is presented in § 4. This is a one hump map obtained by adding a sinusoidal map to the logistic map (both the maps belong to the same universality class). It is found that the scaling index characterizing the behaviour of the Lyapunov exponent near the onset of the chaotic transition is different from that of a single one-hump map. .Section 5 contains the concluding remarks.

2. Modulated logistic map

In this section we briefly review the results obtained by Harikrishnan and Nandakumaran on MLM [9–12]. An MLM is defined by the following pair of equations:

$$X_{n+1} = 4\lambda_n X_n(1 - X_n), \tag{2}$$

$$\lambda_{n+1} = 4\mu\,\lambda_n\,(1 - \lambda_n) \tag{3}$$

with $0 < X_n, \lambda_n, \mu < 1$. Here μ plays the role of the control parameter. This system is analogous to the one suggested by Ruelle although in the present case the time evolution of the λ-system is not slow, but is of the same order as that of the X-system. As μ is varied continuously, the MLM undergoes a sequence of period doubling bifurcations. However, the bifurcation diagram for X is very much different from the one for the λ-system [11].

In order to establish the universality properties of the map we have determined the Feigenbaum constant δ using the method of eigenvalue matching renormalization due to Derrida *et al* [14, 15]. The basic idea of the method is the following. Let us represent the

MLM as

$$(X_{n+1}, \lambda_{n+1}) = T_\mu(X_n, \lambda_n). \tag{4}$$

We linearize the map $T_\mu^{(n)}$ in the neighbourhood of an n-cycle. With each μ we associate a μ' such that the linearization of $T_\mu^{(n)}$ around a point of cycle n is identical to linearization of $T_{\mu'}^{(2n)}$ around a point of cycle $2n$. This gives rise to a relation connecting μ and μ', the fixed point of which is given by μ_∞ universal constant δ is then given by

$$\delta = \left.\frac{\mathrm{d}\mu}{\mathrm{d}\mu'}\right|_\mu. \tag{5}$$

The details of the calculations are presented in [10]. In the first order approximation, we choose $n = 1$, this gives a value of $\delta = 4.4339...$ which is consistent with the Feigenbaum constant for the logistic map. Better agreement can be obtained by choosing higher values for n.

One of the interesting features that we have observed in MLM is that the periodicity of X is identical to that of λ except in the range of values $0.848 < 0.860$ [9, 13] where λ has a periodicity of 2 while X has a periodicity of 4. As μ is varied λ period doubles and so does the variable X. This locking of the periodicities of the two is a special case of the more general situation described by Batra and Varma. For a given value of μ they have derived conditions under which the X-system is in a state of periodicity n or in a state of periodicity which is an integral multiple of n, when the λ-system is in a stable state of periodicity n. For the MLM the two periodicities are identical except in the range of μ mentioned above. This result has an important consequence. It shows that the MLM has a stable n-cycle for the same range of μ for which the logistic map has an n-cycle. Thus the periodicity of X is enslaved to the periodicity of λ. This implies that the ordering of the cycles in MLM is the same as that of the logistic map. Thus the Sarkovskii ordering [16] which is an universal structural property of unimodel maps is maintained in the MLM. Batra and Varma have suggested other possible 2D-systems that may show Sarkovskii ordering. These universal properties of MLM makes it suitable to model physical systems in which the control parameter is time-dependent.

3. Parametrically perturbed logistic map

In this section we consider a logistic map whose control parameter is perturbed by the addition of a periodic term [17]. Instead of changing λ continuously it is changed in a discrete sequence of pulses, repeated periodically, the envelope of the amplitudes of these pulses forming a positive sine profile. Maps where the control parameter is perturbed by a sequence of pulses may be relevant in the study of certain biological systems subjected to periodic stimuli [18]. A logistic system with a similar perturbation but with a cosine profile that includes both the positive and the negative half cycles has been considered earlier [19]. It has been shown that the map undergoes a transition from a fixed point to a state which has the periodicity of the perturbation. We observe that the map has some novel features not considered earlier. Quadrative maps with additive periodic forcing that leads to bistability and the coexistence of multiple attractors have been studied by Sanju and Varma [20].

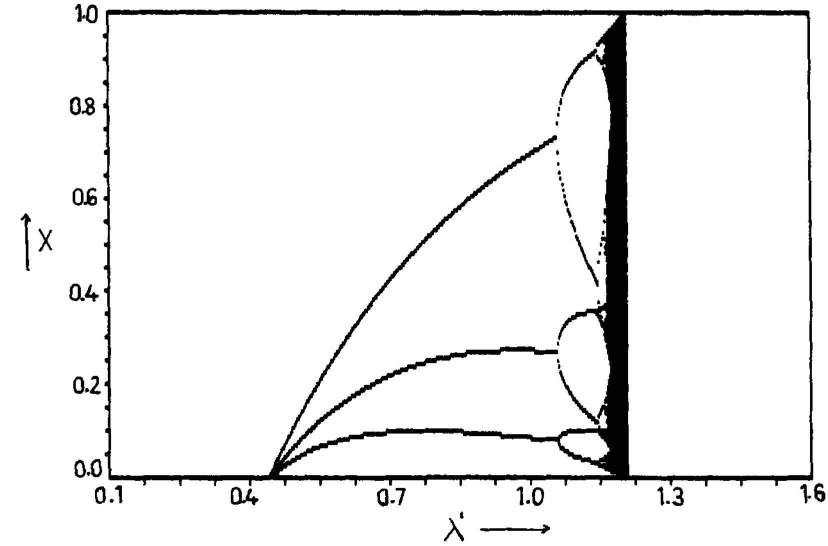

Figure 1. Bifurcation diagram of the parametrically perturbed map for $q = 3$.

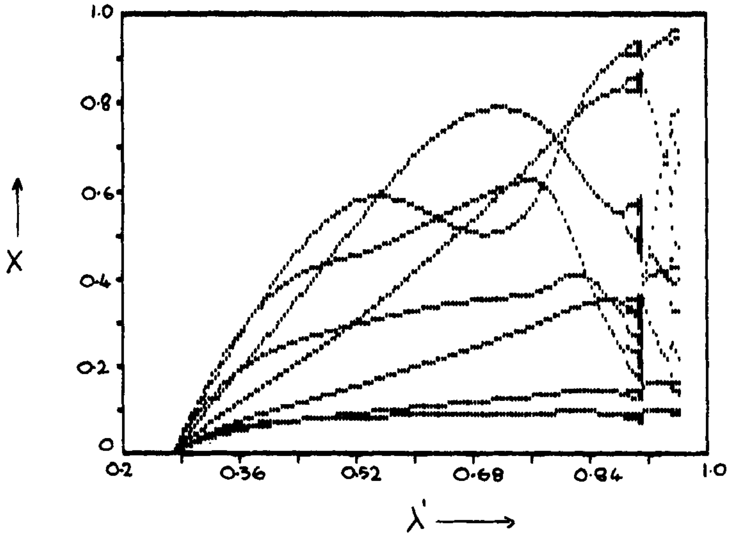

Figure 2. Bifurcation diagram for $q = 8$.

The parametrically perturbed logistic map is defined by the equations

$$X_{n+1} = 4\lambda X_n(1 - X_n),$$

with

$$\lambda = \lambda_0 + \lambda^* \sin(\pi\omega n), \quad (\omega n) \bmod 1. \tag{6}$$

This represents a train of pulses whose periodicity is determined by ω.

If $\omega = p/q$, the control parameter forms a periodic q-sequence. In what follows we chose $q = 1$. ω can also be chosen as irrational. The dynamics of the map (6) is studied

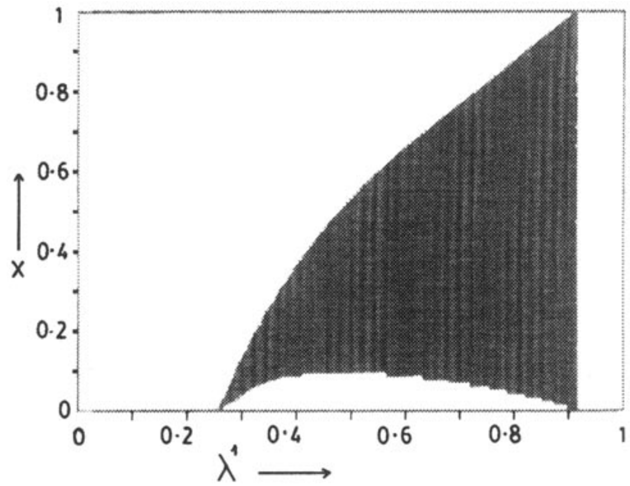

Figure 3. Bifurcation diagram for $q = (5^{1/2} - 1)/2$.

numerically by fixing λ_0 and varying the strength of the perturbation amplitude λ^*. To obtain a wide tunability for λ^* we fixed a low value for $\lambda_0 = 0.1$. For this value of λ_0, the unperturbed logistic map has a stable fixed point $X^* = 0$. The bifurcation diagrams are drawn for $q = 3, 8$ and $(5^{1/2} - 1)/2$ which is the golden mean (figures 1, 2 and 3). For $q = 3$, when λ^* is increased the fixed point $X^* = 0$ becomes unstable and the system makes a transition to a stable 3-cycle and remains locked to it for a very long range of λ^*. After this, period doublings follow in quick succession followed by a chaotic regime. When λ^* is increased still further the iterates leave the interval $(0, 1)$ and escape to infinity. Qualitatively similar features are seen for $q = 8$. However, for $q = 8$, the chaotic region is followed by the reappearance of 8-cycles which further period doubles and goes to chaos. Some of the other novel features observed in this system have not been included in this paper since they form the contents of another paper which has been sent for publication. When $q = (5^{1/2} - 1)/2$, the stable fixed point becomes unstable as λ^* is increased and goes to a quasiperiodic state (figure 3) and remains in that state until escape. When ω is chosen as irrational there are additional interesting phenomena such as strange nonchaotic attractors [21]. However, we do not consider this in the present case.

It is instructive to study the superstable cycles in this map. For superstable cycle, $x = 1/2$ is one of the cycle elements. Metropolis *et al* [5] have studied extensively the sequence of iterates starting from the extremum at $x = 1/2$. These sequences are often called the MSS sequences. The superstable cycles are represented by a sequence of symbols R and L depending on whether the iterate falls on the right (R) or the left (L) of the extremum at $x = 1/2$. We have done a similar analysis for the map (6) and the results are tabulated in table 1 for various q-values. It is clear from the table that the map (6) can support sequences that do not fall under the MSS classification. In fact the first superstable sequence for each ω contains only the symbol L. In the table the sequences indicated by a star belong to the MSS classification.

The parametrically perturbed map is of importance since it may be used to model many experimental situations such as a laser system [22] where the control parameter would be modulated by periodic pulses or pulse trains.

Table 1. Super stable sequences of the parametrically perturbed logistic map. The super stable value of λ^* is indicated by Λ^*.

q		Sequences	Λ^*
3	$k=3$	L^2	0.6672028
	$k=6$	L^2RL^2	1.008131
4	$k=4$	L^3	0.5871376
		$^*RL^2$	0.6724169
		L^2R	0.9041113
5	$k=5$	L^4	0.5268982
6	$k=6$	L^5	0.4899559
		$^*RL^4$	0.5539654
		L^4R	0.7129956
7	$k=7$	L^6	0.4855321
		$^*RL^5$	0.59889335
		L^4R^2	0.6514045
		$^*RL^4R$	0.80803725
8	$k=8$	L^7	0.4585666
		$^*RL^6$	0.5080377
		L^5R^2	0.5843965
		R^3L^4	0.655407
		L^4R^3	0.81519066
		L^4RLR	0.92404916
9	$k=9$	L^8	0.4621206
		RL^7	0.4633298
		R^2L^6	0.5403486
		R^3L^5	0.7149534
		$^*RL^5R^2$	0.720069367
		RL^5R^2	0.80574375
		L^4R^4	0.85825292
		$^*RL^4R^3$	0.9043900
10	$k=10$	L^9	0.44201028
		$^*RL^8$	0.48165978
		L^7R^2	0.508807
		R^3L^6	0.57816015
		L^5R^4	0.691275
		$^*RL^5R^3$	0.7179589
		$RLRL^5R$	0.7626641
		$RLRLRL^4$	0.7773538
		L^4R^2LRL	0.8562051

*Indicates the cycles that occur in the MSS sequences. If k is the period of the cycle the sequence contains $(k-1)$ symbols.

4. Dynamics of a combination map

The combination map is obtained by combining the logistic map with a sinusoidal map. Thus we have a two-parameter one dimensional map given by [23, 24]

$$X_{n+1} = f(X_n, \lambda, A) = 4\lambda X_n(1 - X_n) - A\sin(\pi X_n). \tag{7}$$

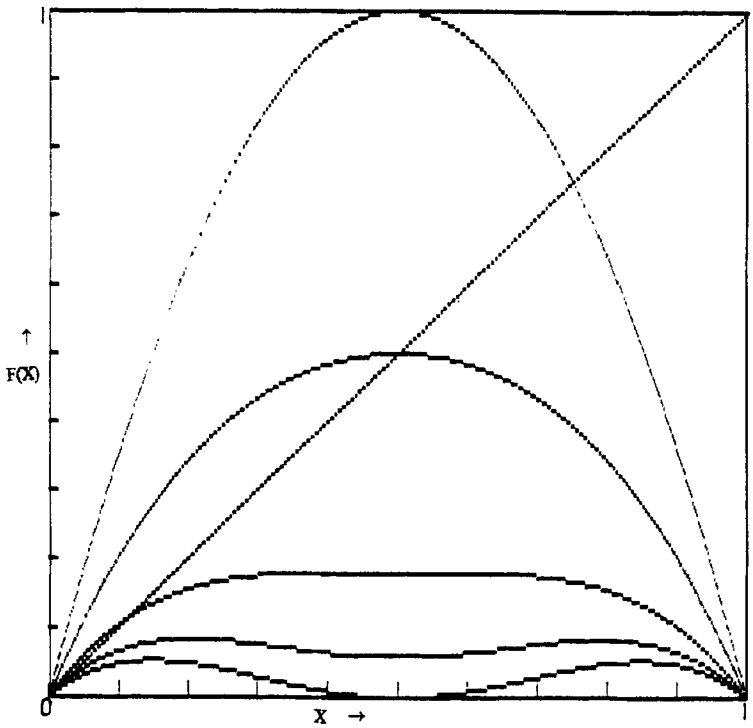

Figure 4. The combination map for $\lambda = 1$. The curves correspond to $A = 0, 0.5, 8/\pi^2, 3/\pi$ and 1 in that order from top to bottom.

On the right hand side both the terms belong to the same universality class, viz, the quadratic family. The motivation for studying the map (7) is to see whether the map still has all the characteristics of the quadratic family. We shall show that it has the same universal constants as the logistic map, however, the scaling index of the Lyapunov exponent at the transition point is different.

The map (7) has an extremum at $x = 1/2$ which is a second order maximum for $(\lambda - 1) < A < 8\lambda/\pi^2$ while it is a minimum for $8\lambda/\pi^2 < A < \lambda$ and there is a point of inflexion at $x = 1/2$ for $A = 8\lambda/\pi^2$. Consequently, $f(X_n, \lambda, A)$ is one humped for $(\lambda - 1) < A < 8\lambda/\pi^2$ and two humped for $8\lambda/\pi^2 < A < \lambda$. It can easily be seen that the parameter A must lie between $(\lambda - 1)$ and λ so as to keep the iterates of (7) within the unit interval (0, 1). The function $f(x, \lambda, A)$ is plotted for various values of A in figure 4. In figure 5, we give the complete bifurcation structure for the map by keeping $\lambda = 1$ and by changing A in steps of 0.001. As the parameter A is slowly tuned, the system retraces the entire period doubling route to chaos in the reverse order and finally settles down to a one-cycle for $A > 0.2435$. One can also consider negative values of A which would bring the system from a state of periodicity to a state of chaos. By changing λ, the value A_∞ of A at which the system makes transition from chaos to order and vice versa, is determined by calculating the Lyapunov exponent and by noting the value of A at which it changes sign. The thick line figure 6 represents the value of A as a function of λ. The lines parallel to it (only two are shown in figure 6) represent the bifurcation lines along which each period doubling occurs. The shaded region corresponds to the chaotic regime of the

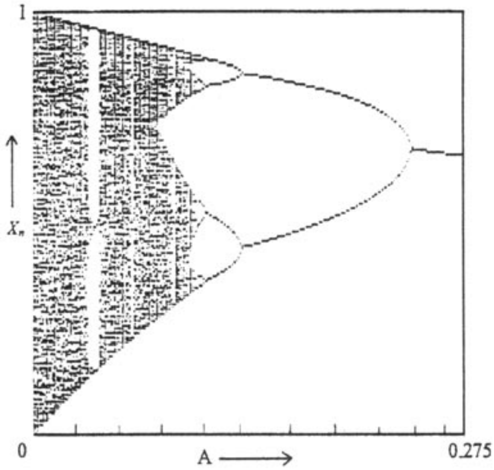

Figure 5. Bifurcation structure of the combination map with $\lambda = 1$.

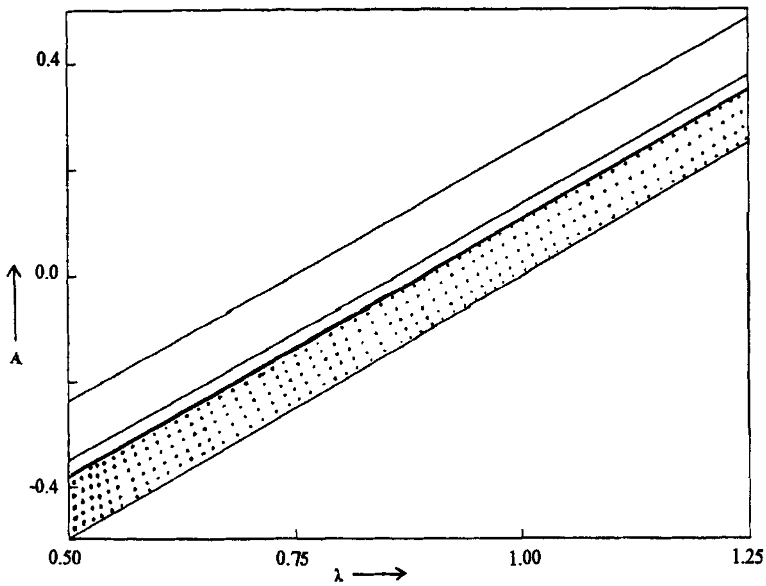

Figure 6. The parameter space (λ, A) of the combination map. The thick line represents the transition from order to chaos while the lines parallel to it are the bifurcation curves.

system where the Lyapunov exponent is generally positive except for windows of periodicity within the chaotic regime. The lowermost line represents the minimum value of A, namely $(\lambda - 1)$ for each λ. By extending this line upwards we can increase the value of λ beyond $\lambda = 1$ by taking suitably high value of A. Also, since the bifurcation curves in the (λ, A) space are parallel lines, it is clear that the Feigenbaum constant δ defined in terms of the bifurcation values A_n for fixed λ must be the same as that for the logistic map alone.

Table 2. The scaling index ν and accumulation point A_∞ for the combination map considered in §4.

λ	A_∞	ν
0.75	−0.1382968450	0.9989838
0.875	−0.0170178750	0.9677731
1.0	−0.1046909450	0.9936844
8	7.2212675805	0.9996790

To see whether the combination map has all the characteristics of the quadratic maximum we have investigated the scaling property of the Lyapunov exponent as a function of $|A - A_\infty|$. For a one-hump map the nature of this scaling near the period doubling accumulation point has been worked out by Hubermann and Rudnick [25]. They have shown that the Lyapunov exponent σ follows the relation

$$\sigma \sim |a - a_\infty|^\nu, \tag{8}$$

where a_∞ is the accumulation point for the period doubling bifurcations and that

$$\nu = \ln 2/\ln \delta, \tag{9}$$

δ being the Feigenbaum constant. The Hubermann–Rudnick (HR) relation (9) indicates that ν depends on z, the order of the map, through the value of δ which is different for maps belonging to different universality classes (different z) [26]. Since δ is the same for the combination map one expects the same value for ν for the combination map also. To test this we have determined ν and A for the combination map and the results are shown in table 2 [24].

It is clear from the table that ν for the combination map is almost double the value of $1/2$ for the quadratic family as predicted by the HR relations and also computed numerically for the logistic map. Thus, for the combination map which also has a quadratic maximum at $x = 1/2$ does not satisfy the HR relations. Numerical studies have shown that a series of bifurcation for the chaotic band take place as $A \to A_\infty$ [23]. However, this cascade of band bifurcations does not take place ad infinitum. This incomplete nature of the cascade for the combination map could be a possible reason for the violation of HR relations for the map.

5. Concluding remarks

In this review we have considered the dynamics of modulated logistic systems. Specifically we have discussed three different systems, viz, the modulated logistic map, the parametrically perturbed logistic map and the combination map obtained by combining two maps in the same quadratic family. Each of these systems shows several interesting features, such as the enslavement of the periodicities (in MLM), occurence of superstable sequences other than the ones included in the MSS classification, the reappearance of periodicity after a chaotic regime (in the parametrically perturbed logistic map) and the violation of the Hubermann–Rudnick scaling relations for the Lyapunov exponents (in the combination map).

Acknowledgements

The author wishes to thank K P Harikrishnan, P P Saratchandran and P R Krishnan Nair for useful discussions.

References

[1] B L Hao, *Chaos – a reprint collection* (World Scientific, Singapore, 1984)
[2] J Peinke, J Parisi, O E Rossler and R Stoop, *Encounter with chaos – Self organised hierarchical complexity in semiconductor experiments* (Springer-Verlag, Berlin-Heidelberg, 1992)
[3] R M May, *Nature (London)*, **261**, 459 (1976)
[4] M J Feigenbaum, *J. Stat. Phys.* **19**, 25 (1978)
[5] N Metropolis, M L Stein and P R Stein, *J. Comb. Theory* **A15**, 25 (1973)
[6] P T Arechi, R Meucci, G Puccioni and J Tredicce, *Phys. Rev. Lett.* **49**, 1217 (1982)
[7] R Kapral and P Mandel, *Phys. Rev.* **A32**, 1076 (1985)
[8] D Ruelle, *Proc. R. Soc. (London)* **A413**, 5 (1987)
[9] K P Harikrishnan and V M Nandakumaran, *Pramana – J. Phys.* **29**, 533 (1987)
[10] K P Harikrishnan and V M Nandakumaran, *Phys. Lett.* **A125**, 465 (1987)
[11] K P Harikrishnan and V M Nandakumaran, *Phys. Lett.* **A133**, 305 (1988)
[12] K P Harikrishnan and V M Nandakumaran, *Phys. Lett.* **A142**, 483 (1989)
[13] Sangeeta Batra and V S Varma, *Pramana – J. Phys.* **37**, 83 (1991)
[14] B Derrida, A Gervois and Y Pomeau, *J. Phys.* **A12**, 269 (1979)
[15] B Derrida and Y Pomeau, *Phys. Lett.* **A80**, 271 (1980)
[16] A N Sarkovskii, *Ukr. Math. J.* **16**, 61 (1964)
[17] P P Saratchandran, V M Nandakumaran and G Ambika, *Proc. of National Seminar on Computational Aspects of Chaos and Nonlinear Dynamics* edited by G Ambika and V M Nandakumaran (Wiley Eastern Ltd., 1994)
[18] S J Schiff, K Jerger, D H Duong, T Chang, M L Spano and W L Ditto, *Nature (London)* **370**, 615 (1994)
[19] M Lucke and Y Saito, *Phys. Lett.* **A91**, 205 (1982)
[20] Sanju and V S Varma, *Phys. Rev.* **E48** 1670 (1993)
[21] J F Heagy and S M Hammel, *Physica* **D70**, 140 (1994)
[22] T Midavainne, D Dangoisse and P Glorieux, *Phys. Rev. Lett.* **55**, 1989 (1985)
[23] P R Krishnan Nair, V M Nandakumaran and G Ambika, *Proc. of the National Seminar on Computational Aspects of Chaos and Nonlinear Dynamics* edited by G Ambika and V M Nandakumaran (Wiley Eastern Ltd., 1994)
[24] P R Krishnan Nair, V M Nandakumaran and G Ambika, *Pramana – J. Phys.* **43**, 421 (1994)
[25] B A Hubermann and J Rudnick, *Phys. Rev. Lett.* **45**, 154 (1980)
[26] G Ambika and K Babu Joseph, *Pramana – J. Phys.* **39**, 193 (1992)