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Effect of a fluctuating parameter mismatch and the associated time-scales on coupled Rossler oscillators

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Abstract. We study the effect of parameter fluctuations and the resultant multiplicative noise on the synchronization of coupled chaotic systems. We introduce a new quantity, the fluctuation rate ϕ as the number of perturbations occurring to the parameter in unit time. It is shown that ϕ is the most significant quantity that determines the quality of synchronization. It is found that parameter fluctuations with high fluctuation rates do not destroy synchronization, irrespective of the statistical features of the fluctuations. We also present a quasi-analytic explanation to the relation between ϕ and the error in synchrony.

Keywords. Chaos; synchronization; parameter mismatch.

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1. Introduction

Synchronization of coupled chaotic systems has generated a lot of research activities over the last several years. Synchronized behaviour has been studied extensively in physical, chemical and biological systems [1–11]. Different types of synchronization such as complete, generalized, lag and phase synchrony are described in literature. One of the methods by which the synchronization of chaotic systems is achieved is by coupling two identical systems, which may be unidirectional or bidirectional. Synchronization in arrays of coupled laser systems has also been investigated under various coupling schemes [8–11]. Complete synchronization of identical chaotic systems is of considerable interest because of its applications in secure communication [10,11]. By identical systems we mean a set of systems whose parameters are exactly equal. It is found that the complete synchronization is not possible when there is a small but finite mismatch of the parameters of the systems [7,12,13]. In coupled non-autonomous systems, phase mismatch or finite constant frequency detuning destroys the synchronization altogether [14].

Though the effect of finite constant parameter mismatch is to destroy synchronization, the effect of a fluctuating parameter mismatch can be different. Such fluctuations can arise either due to the internal instabilities, or due to environment. It is found that spatial parameter fluctuations (constant in time) can give rise to interesting nonlinear phenomenon (for example, occurrence of travelling waves in coupled map lattices, when mismatches do not intersect bifurcation points [15,16] and pattern formation) [17]. In coupled phase oscillators, such fluctuations seem to play the role of pacemakers [18].

Exact synchronization is not usually found in nature, at least as common as phase synchronization. The systems that are designed for applications such as chaotic encryption, will be maintained with identical parameter values, though they are located in different regions in space. However, temporal modifications to the parameter values for which such systems are designed are possible. Here we study the effect of temporal parameter fluctuations with characteristic time-scales on exact synchronization of chaotic systems.

2. Parameter fluctuations

To study the effect of fluctuations it is essential to identify a parameter whose mismatch is most effective in destroying synchronization. Let this parameter be p and the fluctuations to the parameter is assumed to occur in time as follows:

$$\begin{aligned} p_{1t} &= p_0 + \xi_{1t} \\ p_{2t} &= p_0 + \xi_{2t}, \end{aligned} \tag{1}$$

where, ξ_{1t} and ξ_{2t} are two delta correlated zero mean random variables. We define $\widetilde{\Delta p}$, a measure of the amplitude of fluctuations, as

$$\widetilde{\Delta p} = \langle |\delta p(t)| \rangle_t, \tag{2}$$

where $\delta p_t = p_{1t} - p_{2t}$ and $\langle \cdots \rangle_t$ denotes time average. Here we assume that such fluctuations do not intersect the bifurcation points in the parameter space, as suggested in [15].

Parameter fluctuations can be associated with characteristic time-scales. In a laser this can be of the order of nano- or microsecond and in the case of a biological system the time-scales may be of the order of hours or days. To study the effect of time-scales of parameter fluctuation on synchronization, we define the fluctuation rate ϕ, where ϕ = number of perturbations/unit time. Different fluctuation rates can be achieved numerically by modifying the parameter as in eq. (1) only in certain chosen time steps. Rest of the time the value of the parameter remains constant at the modified value. The error in synchrony is studied by varying ϕ.

Coupled Rossler oscillators are well-known for numerical studies in synchronization. This is due to their simplicity, its ability to synchronize and the generalizability of the results to other chaotic systems. We consider a system of bidirectionally coupled Rossler oscillators.

$$\begin{aligned}
\dot{x_1} &= -y_1 - z_1 + c(x_2 - x_1)\\
\dot{y_1} &= x_1 + p_1 y_1\\
\dot{z_1} &= 0.2 + z_1(x_1 - 10)\\
\dot{x_2} &= -y_2 - z_2 + c(x_1 - x_2)\\
\dot{y_2} &= x_2 + p_2 y_2\\
\dot{z_2} &= 0.2 + z_2(x_2 - 10).
\end{aligned} \tag{3}$$

Here, the coupling strength $c = 0.25$, $p_0 = 0.18$ and $\widetilde{\Delta p}$ was fixed to be 0.05 for all fluctuation rates. This selection of parameters is arbitrary, however, variations in these parameters gives qualitatively similar results. Thus we choose values which are best suited for illustrating the concepts. Figure 1 shows the synchronization plot in the presence of parameter fluctuations. It can be seen that the synchronization is robust. With the same value of $\widetilde{\Delta p}$ the synchronization is destroyed with a lower fluctuation rate as shown in figure 2.

The similarity function given by

$$S^2(\Theta) = \frac{\langle [x_1(t+\Theta) - x_2(t)]^2 \rangle}{[\langle x_1^2(t)\rangle \langle x_2^2(t)\rangle]^{1/2}}, \tag{4}$$

is a versatile tool in studying the synchronization properties. In coupled chaotic systems this function can be used to represent the nature of the dynamics in terms of the synchronization error. For example, in [7,12,13] the phase to lag synchronization of coupled systems is described in terms of the variation of $S(\Theta)$ with respect to Θ. It is also useful in studying the dynamics of coupled map lattices [16].

In eq. (4) if Θ is set to zero, gives $S(0)$, the error in synchrony. Figure 3 shows the plot of $S(0)$ vs. fluctuation rate. It can be seen that the error diminishes rapidly with the increase in the fluctuation rate. It is found that a linear relation between

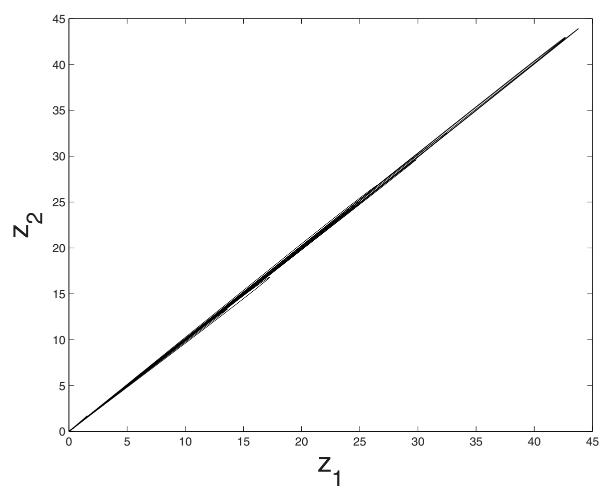

Figure 1. Synchronization is maintained in the presence of parameter fluctuations. $\phi = 1000$ and $\widetilde{\Delta p} = 0.05$.

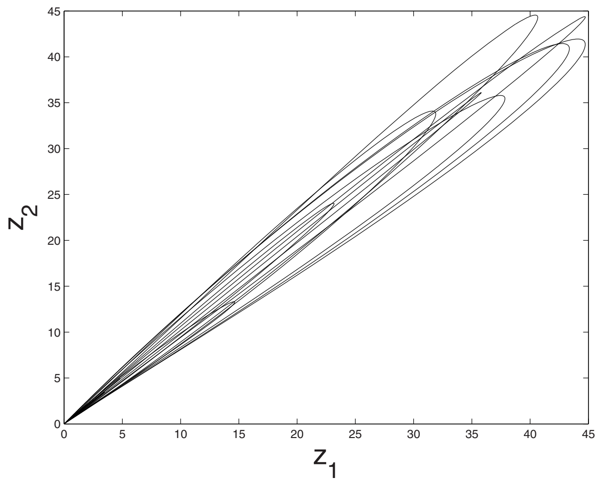

Figure 2. Synchronization is destroyed in the presence of parameter fluctuations with low fluctuation rates. Fluctuation rate $\phi = 25$ and $\widetilde{\Delta p} = 0.05$.

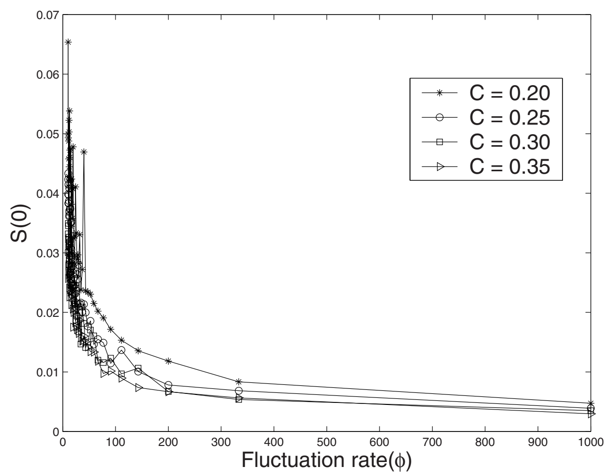

Figure 3. The synchronization error decreases with the increase in the fluctuation rate. It can be seen that high coupling could not stabilize synchronization with lower fluctuation rates. Here $\widetilde{\Delta p} = 0.05$.

the inverse of the fluctuation rate and the error $S(0)$ exists. In figure 4 it is shown that with a linear fit of the form $S(0) = a + b\tau$ where $\tau = \frac{1}{\phi}$ is possible. With $c = 0.9$, a(=0.005) represents the offset value which is much small compared to b(=0.12), which is the slope of the graph, which suggests that $S(0)$ is inversely proportional to ϕ. In addition to random fluctuations to the parameter, we investigated the effect of deterministic modulations to the parameter. The parameters of the coupled systems are assumed to evolve in time as follows:

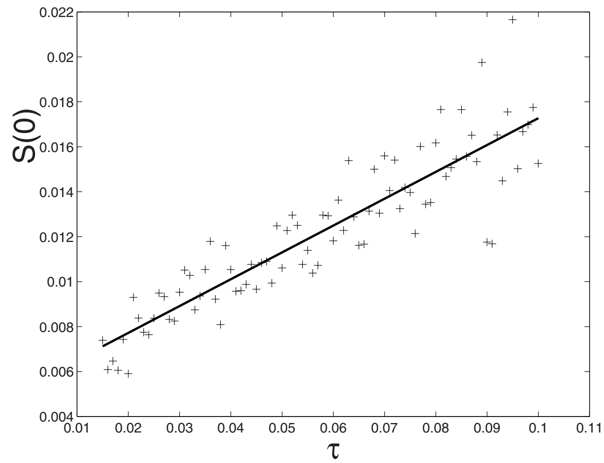

Figure 4. Relation between ϕ and $S(0)$ is found to be of the form $S(0) = a + \ b\tau$, with $c = 0.9$, $a = 0.005$ and $b = 0.12$.

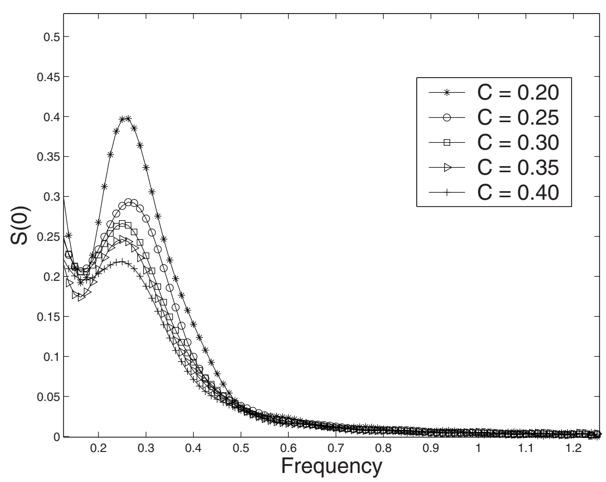

Figure 5. The synchronization error decreases with the increase in the frequency of modulation. It can be seen that the modulation frequency is more important than coupling strength in determining the stability of synchronization. Here the amplitude of modulation $a \ = \ 0.1$.

$$p_1 = p_0 + a \sin 2\pi f t,$$
$$p_2 = p_0 - a \sin 2\pi f t, \tag{5}$$

where f is the frequency of modulation and $a = 0.1$ is the amplitude of modulation. It can be seen in figure 5 that the synchronization error levels off as the frequency of modulation is increased. Note that high coupling strengths only reduce the synchronization error, but the stability of synchrony is achieved only at high modulation frequencies.

In general the robustness of synchronization with high fluctuation rates and destruction of synchronization with low fluctuation rates can be understood as follows: This also applies to other coupled dynamical systems of similar nature. Let the evolution of the coupled systems in phase space be represented by the dynamical equation

$$\dot{X}_1 = f_1(p_1, X_1) + Cf(X_2 - X_1),$$
$$\dot{X}_2 = f_1(p_2, X_2) + Cf(X_1 - X_2), \tag{6}$$

where X represents the phase space variables, p is the parameter whose fluctuation is considered and C is the coupling constant. With eq. (6) we can write an equation for the rate of separation $X_1 - X_2$ of the trajectories as

$$\frac{\mathrm{d}(X_1 - X_2)}{\mathrm{d}t} = \dot{X}_1 - \dot{X}_2 = M(p_1, p_2, X_1, X_2), \tag{7}$$

$M(p_1, p_2, X_1, X_2)$ is a function of the dynamical variables, the parameters of the coupled systems and Δp the parameter mismatch. This can be expanded in terms of Δp and the effect of fluctuations can be separated out.

$$M(p_1, p_2, X_1, X_2) = M_s(p_0, X_1, X_2) + E(p_0, X_1, X_2, \Delta p_1, \Delta p_2), \tag{8}$$

where $E(p_0, X_1, X_2, \Delta p_1, \Delta p_2)$ can be written as

$$E(p_0, X_1, X_2, \Delta p_1, \Delta p_2) = \Delta p_1 \frac{\partial M(p_1, p_2, X_1, X_2)}{\partial p_1}\bigg|_{p_1 = p_0} + \Delta p_2 \frac{\partial M(p_1, p_2, X_1, X_2)}{\partial p_2}\bigg|_{p_2 = p_0}.$$

This is valid for small Δp neglecting its higher powers or if the higher derivatives of $M(p_1, p_2, X_1, X_2, \Delta p_1, \Delta p_2)$ with respect to p is zero. Also it should be noted that the parameter values are not near a bifurcation point. Here $M_s(p_0, X_1, X_2)$ represents the quantity which offers a stable synchronization manifold, that is, when $M_s(p_0, X_1, X_2)$ alone is in the right-hand side of the separation equation, coupled systems synchronize as $t \to \infty$. The conditions for such a synchronization is widely discussed in [19]. The term $E(p_0, X_1, X_2, \Delta p_1, \Delta p_2)$ represents the effect of the parameter mismatch. Coupled systems synchronizes if the overall effect of this term vanishes as $t \to \infty$.

In the present example of coupled systems (eq. (3)) the rate of separation of trajectories can be written as follows:

$$\frac{\mathrm{d}(x_1 - x_2)}{\mathrm{d}t} = (y_1 - y_2) + (z_2 - z_1) + 2C(x_2 - x_1)$$
$$\frac{\mathrm{d}(y_1 - y_2)}{\mathrm{d}t} = (x_1 - x_2) + p_1 y_1 - p_2 y_2$$
$$\frac{\mathrm{d}(z_1 - z_2)}{\mathrm{d}t} = (x_1 z_1 - x_2 z_2) + 10(z_2 - z_1). \tag{9}$$

Assuming that we start from an approximately synchronized state, $x_1 \simeq x_2$, $y_1 \simeq y_2$ and $z_1 \simeq z_2$, we can write eq. (9) as

$$\frac{\mathrm{d}(x_1 - x_2)}{\mathrm{d}t} \simeq 0$$
$$\frac{\mathrm{d}(y_1 - y_2)}{\mathrm{d}t} \simeq p_1 y_1 - p_2 y_2$$
$$\frac{\mathrm{d}(z_1 - z_2)}{\mathrm{d}t} \simeq 0. \tag{10}$$

Here it can be seen that $M(p_1, p_2, X_1, X_2) = p_1 y_1 - p_2 y_2$ and $E(p_0, X_1, X_2, \Delta p_1, \Delta p_2)$ can be calculated as

$$\begin{aligned} E(p_0, X_1, X_2, \Delta p_1, \Delta p_2) &= \Delta p_1 \left.\frac{\partial(p_1 y_1 - p_2 y_2)}{\partial p_1}\right|_{p_1=p_0} \\ &\quad + \Delta p_2 \left.\frac{\partial(p_1 y_1 - p_2 y_2)}{\partial p_2}\right|_{p_2=p_0} \\ &= \Delta p_1 y_1 - \Delta p_2 y_2 \\ &= \xi_{1t} y_1 - \xi_{2t} y_2 \end{aligned} \tag{11}$$

as the instantaneous parameter mismatches $\Delta p_1 = \xi_{1t}$ and $\Delta p_2 = \xi_{1t}$.

Also the form of $E(p_0, X_1, X_2, \Delta p_1, \Delta p_2)$ can be generalized to

$$E(p_0, X_1, X_2, \Delta p_1, \Delta p_2) = \sum_i \alpha_i \xi_{ti} x_i(t) \tag{12}$$

with many coupling schemes. Here ξ_t's are the fluctuation terms, α's are some constants and $x(t)$'s are the phase space variables of the coupled system.

Here the effect of fluctuations vanishes because the ξ_{ti}'s are zero mean rapidly fluctuating quantities and $x(t)$'s are the phase space variables that evolve slowly when compared to the rapid fluctuations or modulations of the parameter. Thus $x(t)$'s can be assumed to be constant, in the time required for the fluctuations to get summed to zero. However, with a low fluctuation rate the quantity $E(p_0, X_1, X_2, \Delta p_1, \Delta p_2)$ can affect synchrony because the phase space evolution time is comparable to the interval where a fixed parameter mismatch persists. Thus, with a lower fluctuation rate, the system always gets time to respond to the parameter mismatch before being canceled out. In other words, the sum in the RHS of eq. (12) does not vanish without considerably modifying the phase space variables $x(t)$ when the fluctuation rates are low. Apart from Gaussian random fluctuations, we studied perturbations with a uniform distribution. The results were qualitatively similar to that of Gaussian perturbations. This suggests that the fluctuation rate ϕ is more significant than the statistical nature of the fluctuations.

In a case where the parameters are modulated, the fluctuating terms ξ's are replaced by the oscillating terms. The mechanism of retaining stability of synchrony at high frequency modulations is similar to that of random parametric perturbations. The contributions of the parameter mismatch vanishes with fast zero mean oscillations. The analytical treatment of this is straightforward and little different from that of random fluctuations.

The effect of noise on synchronization has been studied in the past. In most of the cases, noise reduces the quality of synchrony or destroys synchronization [20,21]. Interestingly, there are also cases where synchronization is robust to noise [22] or even induce synchronization [23].

A comparison of additive noise and parameter fluctuation in view of the fluctuation rates is interesting. Though noise and parameter fluctuations affect synchronization, their effects on the dynamics are not essentially the same. Noise induces perturbations to the phase space variables that decay as the system evolves and more. In a case where the parameter fluctuates, the resultant perturbations do not die out with the evolution of the system. It remains the same until it is corrected manually or the fluctuations modify the parameter to a new value or in other words the mismatches has no dynamical evolution. In actual physical systems the lifetime of a modified parameter value may follow a statistical distribution which is unique to the system. However, once a fluctuation rate is defined, much of our findings will be relevant.

In conclusion, we have studied the effect of parameter fluctuations on the synchronization of coupled chaotic systems. It is found that the most significant entity that determines the quality of synchronization is the fluctuation rates. Also it is observed that the time-scales with which the parameter fluctuates is more significant than the statistical or mathematical features of the fluctuations. Parameter fluctuations or modulations may also have much higher significance in coupled arrays of nonlinear oscillators and in biological systems which exhibit synchronized behaviour. We hope that our studies will give a motivation in this direction.

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