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Hopf bifurcation in parallel polarized Nd:YAG laser

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ABSTRACT

Dynamics of Nd:YAG laser with intracavity KTP crystal operating in two parallel polarized modes is investigated analytically and numerically. System equilibrium points were found out and the stability of each of them was checked using Routh–Hurwitz criteria and also by calculating the eigen values of the Jacobian. It is found that the system possesses three equilibrium points for (I_j, G_j), where $j = 1, 2$. One of these equilibrium points undergoes Hopf bifurcation in output dynamics as the control parameter is increased. The other two remain unstable throughout the entire region of the parameter space. Our numerical analysis of the Hopf bifurcation phenomena is found to be in good agreement with the analytical results. Nature of energy transfer between the two modes is also studied numerically.

1. Introduction

Laser systems have been the all time favorite of researchers because of the rich variety of dynamics they exhibit. A lot of research has been carried out to study the chaotic fluctuations in the output dynamics of laser systems. Among them, instabilities in multimode solid state lasers were of special interest. Intracavity doubled continuous wave infrared lasers are efficient sources of coherent visible light. Nd:YAG lasers can be developed as cw visible sources using high power laser diode arrays for pumping and also doubling crystals with large nonlinear gain coefficient. It was Baer [1] who first reported large amplitude fluctuations in this laser system. He observed that large amplitude fluctuations and longitudinal mode instabilities arise in the output of diode pumped Nd:YAG laser in the presence of an intracavity doubling crystal. Coupling of various longitudinal modes of the laser by sum frequency generation was found to be the origin of these instabilities. He developed a deterministic rate equation model to explain these fluctuations. Using this model, he was able to predict the dependence of these fluctuations on the pump level, the nonlinear coupling constant and the number of oscillating modes.

A detailed analysis of the periodic and the chaotic fluctuations in the output intensity of multimode solid state laser was carried out by Bracikowski and Roy [2]. They studied the Nd:YAG laser with intracavity KTP crystal and obtained several interesting results. It is possible to eliminate the chaotic fluctuations in this laser and also obtain complex periodic waveforms such as antiphase states by varying the relative orientation of the YAG and KTP crystals. They have also made a statistical study of these chaotic fluctuations.

Studies by Oka and Kubota have shown that the laser dynamics is strongly influenced by the polarizations of the cavity modes [3]. This dependence is due to the fact that the amount of green light produced by sum frequency generation depends on whether the contributing fundamental modes are polarized parallel or orthogonal to each other. It has been shown that when the laser is operating with two orthogonally polarized modes the chaotic fluctuations in the output intensity can be stabilized through a reverse period doubling bifurcation by varying a particular control parameter (relative orientation of

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Table 1
Parameter values used in the numerical simulation of Nd:YAG laser operating with two parallel polarized modes.

τ_c, cavity round trip time	0.2 ns
τ_f, fluorescence life time	240 μs
α, cavity loss parameter	0.01
β, cross-saturation parameter	0.7
ε, nonlinear gain coefficient	5×10^{-6}
γ, small signal gain related to pump rate	0.05

YAG and KTP crystal) [4]. An entirely different dynamical behavior is expected if the modes chosen are having the same polarization state. In this paper, we study the dynamics of Nd:YAG laser with intracavity KTP crystal operating in two parallel polarized modes analytically and numerically. It is found that the laser output exhibits Hopf bifurcation as the control parameter is varied. A detailed stability analysis of the bifurcation phenomena is carried out by using Routh–Hurwitz criteria [5–8] and also by calculating the eigen values of the Jacobian. Effect of change in orientation of YAG and KTP crystal on the energy transfer between the two laser cavity modes is also studied [4].

In Section 2, we describe the laser model used in our numerical analysis which is Nd:YAG laser operating with two parallel polarized modes. Analytical and numerical results are discussed in Section 3. Stability analysis of equilibrium points are carried out using Routh–Hurwitz criteria. Hopf bifurcation phenomena in the output dynamics is verified numerically. Energy exchange between the modes is also studied in this section. In Section 4, we give the conclusions.

2. Laser model

For numerical work, we consider an Nd:YAG laser with intracavity KTP crystal. Nd:YAG laser usually lases at 1064 nm in the infrared. The laser output is stable without the intracavity doubling crystal. However, when a nonlinear KTP crystal is inserted into the laser cavity, some of the infrared fundamentals are converted into green light (532 nm) by the process of second harmonic generation and sum frequency generation. Different longitudinal modes get coupled due to this sum frequency generation, which again produces deterministic intensity fluctuations in the laser output. Therefore, with the crystal inserted into the laser cavity, the output intensity exhibits periodic and chaotic fluctuations. The system can be modeled by the rate equations for the intensity I_k and gain G_k for the kth longitudinal mode [1].

$$\tau_f \frac{dG_k}{dt} = \gamma - \left(1 + I_k + \beta \sum_{j \neq k} I_j\right) G_k, \tag{1}$$

$$\tau_c \frac{dI_k}{dt} = \left(G_k - \alpha - g\varepsilon I_k - 2\varepsilon \sum_{j \neq k} \mu_{jk} I_j\right) I_k, \tag{2}$$

$k = 1, 2, 3, \ldots$ are the mode numbers.

In Eqs. (1) and (2) τ_c is the cavity round trip time, τ_f is the fluorescence life time, α is the cavity loss parameter, β is the cross-saturation parameter, ε is the nonlinear gain coefficient and γ is the small signal gain related to pump rate. The parameter values used in the numerical simulation are given in Table 1 [2]. g is a geometrical factor whose value depends on the angle between the YAG and KTP fast axes as well as on the phase delays due to their birefringence.

The amount of green light produced by the sum frequency generation depends on whether the contributing longitudinal modes are polarized parallel or orthogonal to each other. In Eq. (2), for modes having same polarization as the kth mode, $\mu_{jk} = g$, while $\mu_{jk} = (1 - g)$ for modes having orthogonal polarization. This difference is due to the different amounts of sum frequency generated green light produced by pairs of parallel polarized modes or by pairs of orthogonally polarized modes [2].

3. Results

In this section, we present the results of our analytical and numerical studies on the dynamics of Nd:YAG laser operating in two parallel polarized modes.

It has been proved earlier that for Nd:YAG laser operating in two orthogonally polarized modes the output intensity fluctuations change from chaotic to stable behavior through a reverse period doubling bifurcation sequence as the system control parameter is continuously varied. Since there is a strong dependence of output dynamics on the polarizations of the laser cavity modes, an entirely different dynamical behavior is expected for a laser operating in two parallel polarized modes.

The rate equations for intensity and gain (Eqs. 1 and 2) are solved using Matlab to get analytical expressions for the solutions in terms of the control parameter g. The system fixed points are obtained by substituting different g values in these expressions. It is found that the laser system has got nine fixed points as there are nine sets of solutions for (I_j, G_j), where $j = 1, 2$. Out of them only three sets are having real valued solutions; one with equal values for I_1 and I_2; the other two with either I_1 or I_2 equal to zero.

3.1. Stability analysis

To determine the stability of fixed points Eqs. (1) and (2) are linearized around the fixed points. Then we get the equation governing the evolution of any small perturbation $(\delta I_1, \delta G_1, \delta I_2, \delta G_2)$ around the steady state $(I_1^*, G_1^*, I_2^*, G_2^*)$

$$\begin{pmatrix} \delta \dot{I}_1 \\ \delta \dot{G}_1 \\ \delta \dot{I}_2 \\ \delta \dot{G}_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 & 0 \\ D_1 & E_1 & F_1 & 0 \\ C_2 & 0 & A_2 & B_2 \\ F_2 & 0 & D_2 & E_2 \end{pmatrix} \begin{pmatrix} \delta I_1 \\ \delta G_1 \\ \delta I_2 \\ \delta G_2 \end{pmatrix}, \tag{3}$$

where

$$A_1 = \frac{G_1 - \alpha - 2 * g * \varepsilon * I_1 - 2 * g * \varepsilon * I_2}{\tau_c},$$

$$B_1 = \frac{I_1}{\tau_c},$$

$$C_1 = \frac{-2 * g * \varepsilon * I_1}{\tau_c},$$

$$D_1 = \frac{-G_1}{\tau_f},$$

$$E_1 = \frac{-1 - I_1 - \beta * I_2}{\tau_f},$$

$$F_1 = \frac{-\beta * G_1}{\tau_f},$$

$$A_2 = \frac{G_2 - \alpha - 2 * g * \varepsilon * I_2 - 2 * g * \varepsilon * I_1}{\tau_c},$$

$$B_2 = \frac{I_2}{\tau_c},$$

$$C_2 = \frac{-2 * g * \varepsilon * I_2}{\tau_c},$$

$$D_2 = \frac{-G_2}{\tau_f},$$

$$E_2 = \frac{-1 - I_2 - \beta * I_1}{\tau_f},$$

$$F_2 = \frac{-\beta * G_2}{\tau_f}.$$

Hence, the Jacobian of the system at the steady state $(I_1^*,\ G_1^*,\ I_2^*,\ G_2^*)$ is given by

$$J = \begin{pmatrix} A_1 & B_1 & C_1 & 0 \\ D_1 & E_1 & F_1 & 0 \\ C_2 & 0 & A_2 & B_2 \\ F_2 & 0 & D_2 & E_2 \end{pmatrix}. \tag{4}$$

The characteristic equation of the system is given as

$$\begin{vmatrix} A_1 - \lambda & B_1 & C_1 & 0 \\ D_1 & E_1 - \lambda & F_1 & 0 \\ C_2 & 0 & A_2 - \lambda & B_2 \\ F_2 & 0 & D_2 & E_2 - \lambda \end{vmatrix} = 0, \tag{5}$$

where λ is the eigen value of J.

Eq. (5) can be rewritten as

$$a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{6}$$

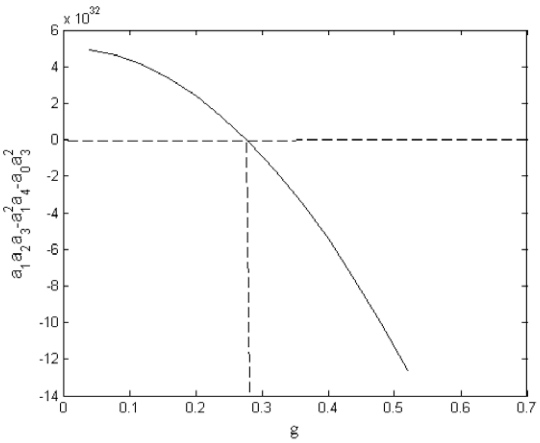

Fig. 1. Graph showing the variation of the coefficient $(a_1a_2a_3 - a_1^2a_4 - a_0a_3^2)$ with g value for solutions with equal values for I_1 and I_2.

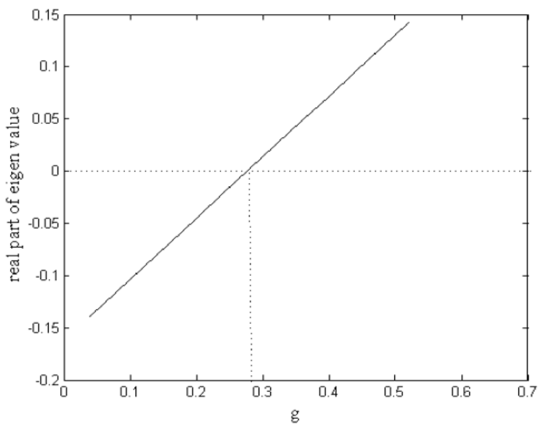

Fig. 2. Graph showing the variation of real part of eigen value with g for solutions with equal values for I_1 and I_2.

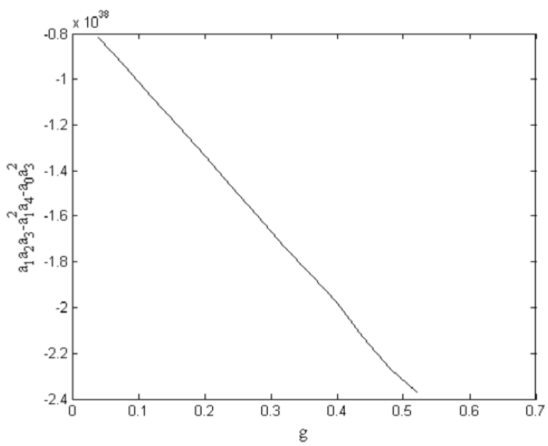

Fig. 3. Graph showing the variation of the coefficient $(a_1a_2a_3 - a_1^2a_4 - a_0a_3^2)$ with g value for solutions with either I_1 or I_2 equal to zero.

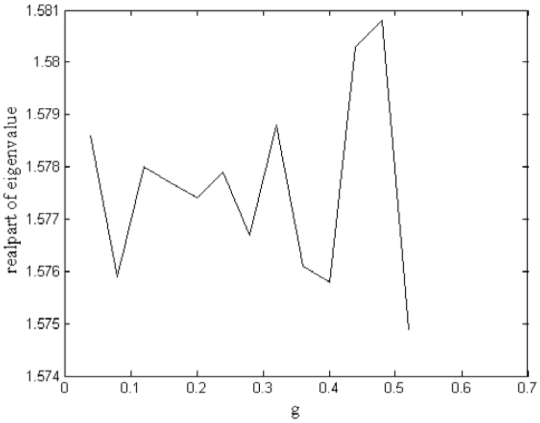

Fig. 4. Graph showing the variation of real part of eigen value with g for solutions with either I_1 or I_2 equal to zero.

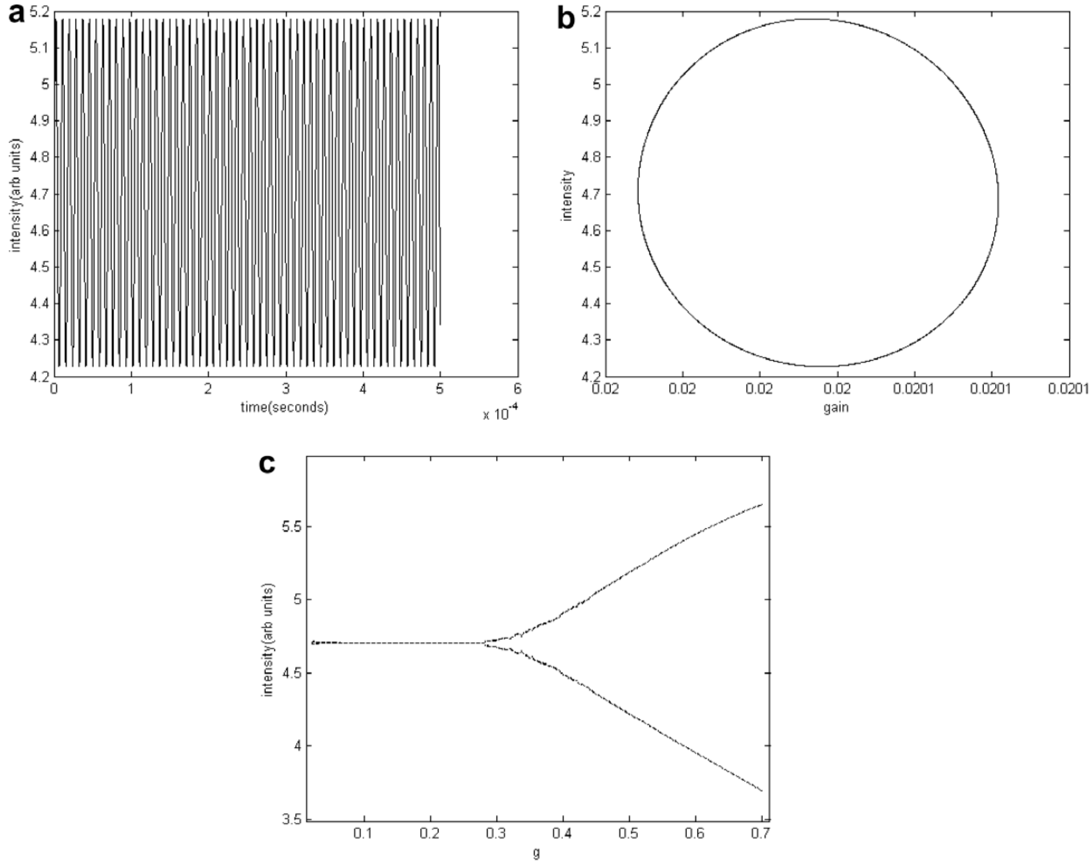

Fig. 5. (a) Intensity time series plot for the laser at $g = 0.5$. (b) Phase space plot for the laser at $g = 0.5$. (c) Bifurcation diagram where the maxima and minima of total output intensity is plotted as a function of the control parameter g.

where

$$
\begin{aligned}
a_4 &= 1,\\
a_3 &= -A_1 - E_1 - E_2 - A_2,\\
a_2 &= A_1E_1 + (A_1 + E_1)(A_2 + E_2) + (A_2E_2 - B_2D_2) - B_1D_1 - C_1C_2,\\
a_1 &= C_1C_2E_1 - B_1F_1C_2 - B_2F_2C_1 + C_1C_2E_2 + (A_2 + E_2)(B_1D_1 - A_1E_1) - (A_2E_2 - B_2D_2)(A_1 + E_1),\\
a_0 &= (A_1E_1 - B_1D_1)(A_2E_2 - B_2D_2) - B_1B_2F_1F_2 + B_1F_1C_2E_2 + C_1E_1B_2F_2 - C_1C_2E_1E_2.
\end{aligned}
$$

According to Routh–Hurwitz criteria the stability conditions are [6]

$$\begin{aligned}
&a_0 > 0,\\
&a_3 > 0,\\
&a_2a_3 - a_1a_4 > 0,\\
&a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 > 0.
\end{aligned} \tag{7}$$

We have checked the stability of our solutions based on this criterion.

(1) Solutions with equal values for I_1 and I_2.

The control parameter is increased and the stability conditions are checked for each case. It is found that the attractor is a stable fixed point for small g values. All the stability conditions are satisfied in this region. At $g = 0.28$,

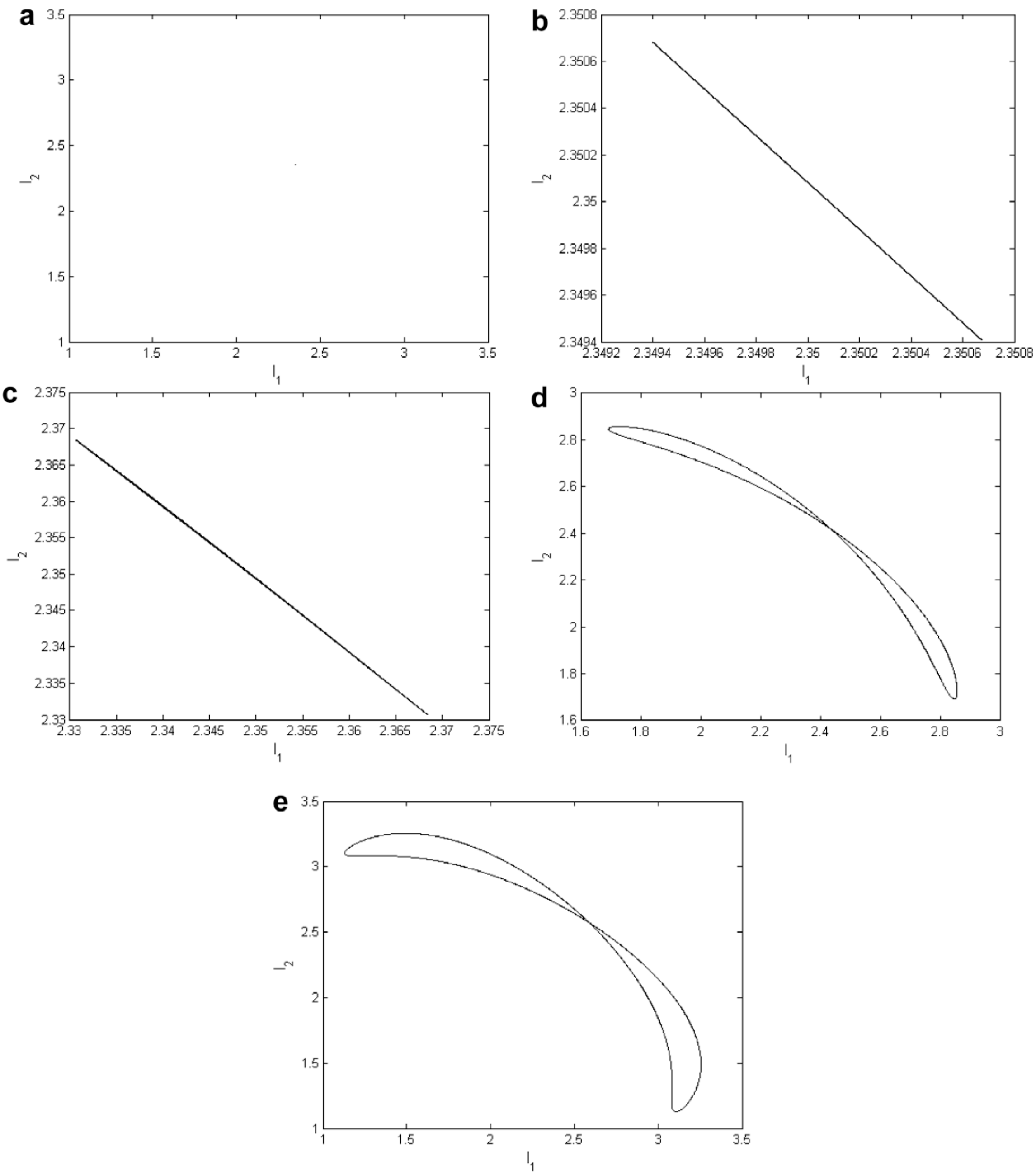

Fig. 6. (a) Intensity in the X polarized mode plotted against that in the Y polarized mode for (a) $g = 0.08$ (b) $g = 0.28$ (c) $g = 0.32$ (d) $g = 0.4$ (e) $g = 0.5$.

$a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 < 0$, which is a clear indication of the loss of stability of the fixed point. Thus, the fixed point loses stability at $g = 0.28$ and becomes a limit cycle, which remains stable for higher g values. Fig. 1 shows the variation of the coefficient $(a_1a_2a_3 - a_1^2a_4 - a_0a_3^2)$ with g value.

We have also calculated the eigen values for the characteristic polynomial for every g value. At $g = 0.28$, real part of two eigen values become positive, which again confirms the loss of stability of the fixed point. Variation in the real part of eigen value is plotted as a function of g in Fig. 2.

This phenomenon where fixed point losses stability at a particular value of the control parameter and changes into a limit cycle is called Hopf bifurcation.

(2) Solutions with either I_1 or I_2 equal to zero.

Stability analysis of these fixed points is also carried out as in previous case. It is found that the two fixed points remain unstable for the entire range of g values. Fig. 3 is a plot showing the variation of coefficient $(a_1a_2a_3 - a_1^2a_4 - a_0a_3^2)$ with g value for these solutions. The coefficient remains negative for the entire range of g value which again shows the unstable nature of these solutions. Real part of eigen values is plotted as a function of g in Fig. 4. Real part of eigen values are found to be positive for every g value which show that the solutions are Routh–Hurwitz unstable.

Eqs. (1) and (2) are integrated numerically using Runge–Kutta fourth-order method for different g values. Fig. 5a shows the intensity time series plots for the laser output for $g = 0.5$, where we can see oscillations with constant amplitude which remains in that state for higher g values. Fig. 5b is the phase space plot for the laser where the total output intensity is plotted against total gain for $g = 0.5$. We can see the phase trajectory evolving as a clear limit cycle for this g value. The Hopf bifurcation phenomena is evident from the bifurcation diagram where the maxima and minima of total output intensity is plotted against the control parameter (Fig. 5c).

We also study the exchange of energy between the two modes. For this the intensity in the X polarized mode is plotted against that in the Y polarized mode. Fig. 6 shows the plots for different values of g. In the stable region for $g = 0.08$, it is a single spot indicating that there is no exchange of energy between the modes (Fig. 6a). For $g = 0.28$ and $g = 0.32$, it is a straight line which implies that a linear relationship exists between the energy of two modes in this region (Fig. 6b and c). It also shows that the total energy of two modes remain a constant in this region. As the g value is increased it changes into a closed loop (corresponding to the limit cycle region) indicating a periodic exchange of energy between the two modes (Fig. 6d and e).

4. Conclusions

Nonlinear dynamics of Nd:YAG laser with intracavity KTP crystal operating with two parallel polarized modes is studied. System equilibrium points were found out analytically. It was found that the system possesses three equilibrium points. The stability of each of them was investigated using Routh–Hurwitz criteria and also verified by calculating the eigen values of the Jacobian. One of the equilibrium points loses stability at a particular value of the control parameter ($g = 0.28$) and evolves as a limit cycle – the phenomenon called Hopf bifurcation. The other two equilibrium points remain unstable through out the entire region of g value. This is different from the case where reverse period doubling route from chaos to stability was found for laser oscillating with two orthogonally polarized modes. Occurrence of Hopf bifurcation in the output dynamics is investigated numerically also.

Energy sharing between the two longitudinal modes is studied as a function of the control parameter. Polarization of the cavity modes is found to have great influence on the nature of energy transfer between the modes.

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